Department of Electrical and Computer Engineering

ELECTENG 721 Radio Systems
ELECTENG 737 Advanced Radio Systems

Waveguides

Andrew Austin

1 Introduction

Waveguide is a form of transmission line that is very commonly used in the radio-frequency (RF), microwave and millimeter-wave bands[1] to carry signals and power. Waveguide is described by the shape of the cross-section, and typically these are *rectangular* or *circular*. While most waveguides at RF are made from metal, dielectric waveguides are also possible, particularly at higher frequencies (e.g., optical fibre), and curved sections can also be fabricated. In this module we will restrict our analysis to straight, rectangular metal waveguide. As with all other electromagnetic problems, waveguides are subject to Maxwell's Equations (and the corresponding boundary conditions), however, unlike most, metal waveguides have exact *analytical* solutions. We will start by considering the electric and magnetic fields formed by a plane wave incident on a perfectly conducting boundary.

2 Wave Reflection at a Perfectly Conducting Boundary

Fig. 1 shows a plane wave incident on a perfect electric conductor (PEC), with angle of incidence θ_i. The wave is assumed to have *perpendicular polarisation*, i.e., the electric field, **E**, is perpendicular to the *plane of incidence*. (The plane of incidence is the plane that contains the incident ray and the normal to the surface). This polarisation is also termed 'horizontal'

[1] While we will not consider *optical fibre* in this course, it is also a form of waveguide.

version: acma 2018.1

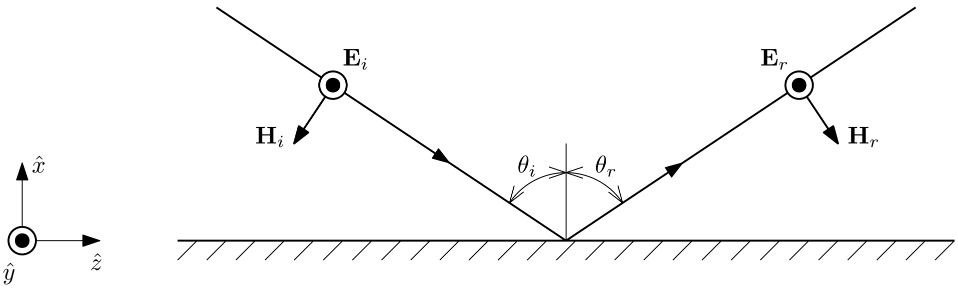


Fig. 1: Perpendicular polarised plane wave incident on a PEC surface.

polarisation[2].

2.1 Incident Wave

In Fig. 1, the incident and reflected plane waves (when considered separately) are transverse electromagnetic waves, i.e., there are no electric or magnetic field components in the direction of propagation. An expression for the electric field of the incident wave[3] in Fig. 1 is

$$\mathbf{E}_i(x,z) = E_0 \exp\left\{-jk\left(-x\cos\theta_i + z\sin\theta_i\right)\right\}\hat{a_y}, \tag{1}$$

where E_0 represents the magnitude of the incident wave, and k is the wavenumber, given by $k = \dfrac{2\pi}{\lambda}$. The corresponding expression for the magnetic field is given by

$$\mathbf{H}_i(x,z) = \left[-\frac{E_0}{\eta_0}\sin\theta_i\hat{a_x} - \frac{E_0}{\eta_0}\cos\theta_i\hat{a_z}\right]\exp\left\{-jk\left(-x\cos\theta_i + z\sin\theta_i\right)\right\}, \tag{2}$$

where η_0 is the intrinsic impedance of free space.

2.2 Reflected Wave

Expressions for the electric and magnetic fields of the reflected wave in Fig. 1 are

$$\mathbf{E}_r(x,z) = E_r \exp\left\{-jk\left(+x\cos\theta_r + z\sin\theta_r\right)\right\}\hat{a_y}, \tag{3}$$

$$\mathbf{H}_i(x,z) = \left[-\frac{E_r}{\eta_0}\sin\theta_r\hat{a_x} + \frac{E_r}{\eta_0}\cos\theta_r\hat{a_z}\right]\exp\left\{-jk\left(+x\cos\theta_r + z\sin\theta_r\right)\right\}, \tag{4}$$

where E_r is the magnitude of the reflected wave. Note that we have not yet made any assumptions for the reflection coefficient, (1)–(4) are simply the expressions for the field components for two perpendicularly polarised TEM waves.

[2] The other polarisation that is possible is when the electric field is parallel to the plane of incidence (also called vertical polarisation).

[3] This is only a function of x and z, as we are assuming a plane wave, and hence the field is constant in the y direction.

2.3 Total Fields

As electric and magnetic field are linear, the total field is the superposition of the incident and reflected, i.e.,

$$\mathbf{E}_t = \mathbf{E}_i + \mathbf{E}_r, \tag{5}$$

$$\mathbf{H}_t = \mathbf{H}_i + \mathbf{H}_r. \tag{6}$$

Note that the fields are vectors and have to be added vectorially. On a PEC boundary, the component of the electric field tangential to the surface must be zero. In Fig. 1, the PEC boundary is an infinite two-dimensional plane (lying in the $\hat{y} - \hat{z}$ plane at $x = 0$). The $\hat{y}$ component of the electric field is tangential to this surface, and thus

$$E_0 \exp\left\{-jkz\sin\theta_i\right\} + E_r \exp\left\{-jkz\sin\theta_r\right\} = 0, \tag{7}$$

for all z. This thus requires $\theta_i = \theta_r = \theta$ and $E_r \equiv -E_0$. Substituting (1) and (3) into (5), yields

$$\begin{aligned}\mathbf{E}_t(x,z) &= E_0\left(\exp\left\{jkx\cos\theta\right\} - \exp\left\{-jkx\cos\theta\right\}\right)\exp\left\{-jkz\sin\theta\right\}\hat{a_y} && (8)\\ &= E_0 2j\sin\left(kx\cos\theta\right)\exp\left\{-jkz\sin\theta\right\}\hat{a_y}. && (9)\end{aligned}$$

We can write a similar expression for the total magnetic field,

$$\begin{aligned}\mathbf{H}_t(x,z) = &-\frac{E_0}{\eta_0}\sin\theta\ 2j\sin\left(kx\cos\theta\right)\exp\left\{-jkz\sin\theta\right\}\hat{a_x}\\ &-\frac{E_0}{\eta_0}\cos\theta\ 2j\cos\left(kx\cos\theta\right)\exp\left\{-jkz\sin\theta\right\}\hat{a_z}.\end{aligned} \tag{10}$$

Consider the total electric field (9):

- In the $\hat{z}$ direction, the field has the characteristic of a travelling wave, i.e.,

  $$\exp\left\{-jkz\sin\theta\right\} \equiv \exp\left\{-j\beta z\right\}. \tag{11}$$

  However, as $\theta \to 0 \quad \exp\left\{-jkz\sin\theta\right\} \to 1$, implying no propagation (as we would expect for normal incidence). As the field has a magnetic field component in $\hat{z}$, this is *not* a TEM wave.

- In the $\hat{x}$ direction, the field has the characteristic of a standing wave, i.e.,

  $$|E_y| \propto \sin\left(kx\cos\theta\right), \tag{12}$$

  as depicted in Fig. 2.

Fig. 2 shows that the total electric field goes to zero at distances nd (for $n = 1\ldots$) above the PEC surface. At these locations we could place *another* PEC plane without any effect on the fields. If d is the distance between successive minima, then

$$kd\cos\theta = \pi \tag{13}$$

and hence

$$\begin{aligned}d &= \frac{\pi}{k\cos\theta}\\ &= \frac{\lambda}{2\cos\theta} \geq \frac{\lambda}{2}\end{aligned} \tag{14}$$

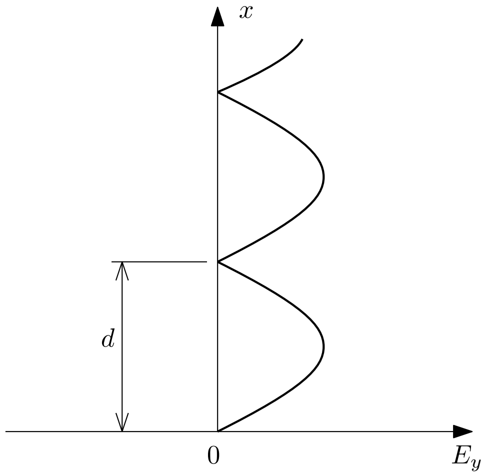


Fig. 2: Electric field magnitude above a PEC surface.

2.4 Field Visualisations

Fig. 3(a) shows the real part[4] of the incident electric field for a TEM wave above a PEC ground plane (at $x = 0$ m). The operating frequency for this wave is 1.0 GHz and the angle of incidence is $\theta = 45°$. Similarly, Fig. 3(b) shows the real part of the reflected wave. Fig. 3(c) shows the resulting total field formed by the superposition of the incident and reflected waves. In Fig. 3(c) we can clearly observe the field goes to zero at $d = 0.212$ m, as expected from (14).

3 Parallel Plate Transmission Line

Fig. 4 shows a 'transmission line' formed by two infinite metal plates. Assuming propagation in the $+\hat{z}$ direction, we know a TEM solution would have

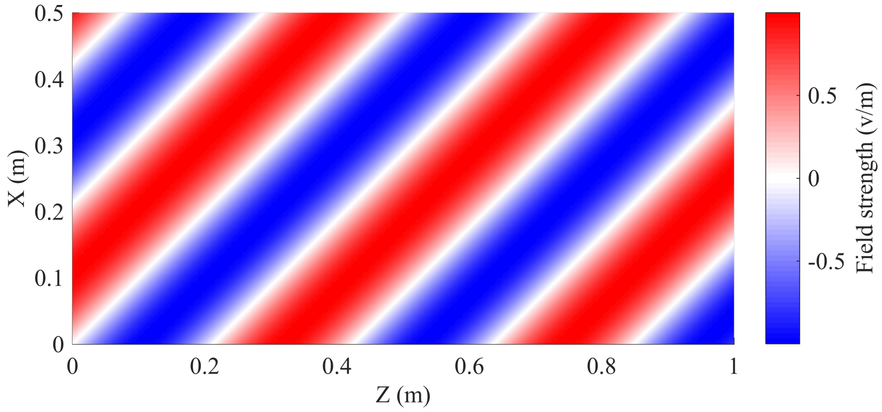
$$\mathbf{E} = E_0 \exp\{-jkz\}\, \hat{a_x} \tag{15}$$

$$\mathbf{H} = \frac{E_0}{\eta_0} \exp\{-jkz\}\, \hat{a_y}. \tag{16}$$

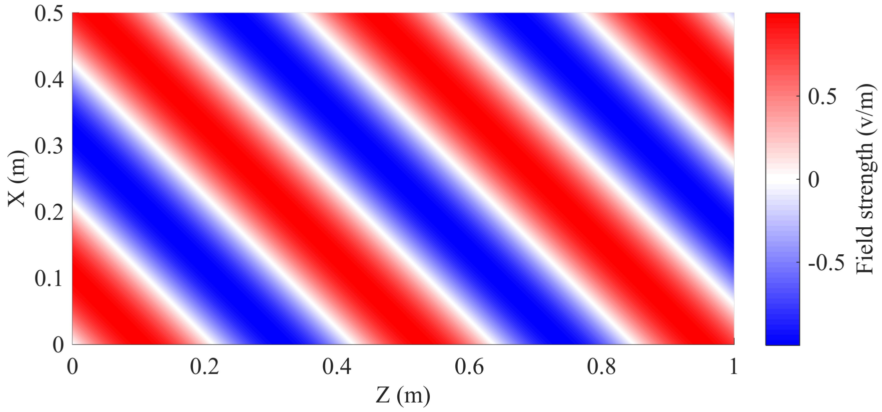
However, if $a \geq \frac{\lambda}{2}$, there is another possibility arising, because it is always possible to find θ such that $\frac{\lambda}{2\cos\theta} = a$ and thus obtain reflections from the walls. This "new" solution has one electric field component E_y and two magnetic field components, H_x *and* H_z, i.e., unlike TEM it has a field component in the direction of propagation. This is a transverse electric (TE) mode.

On the walls of the parallel plate transmission line the tangential com-

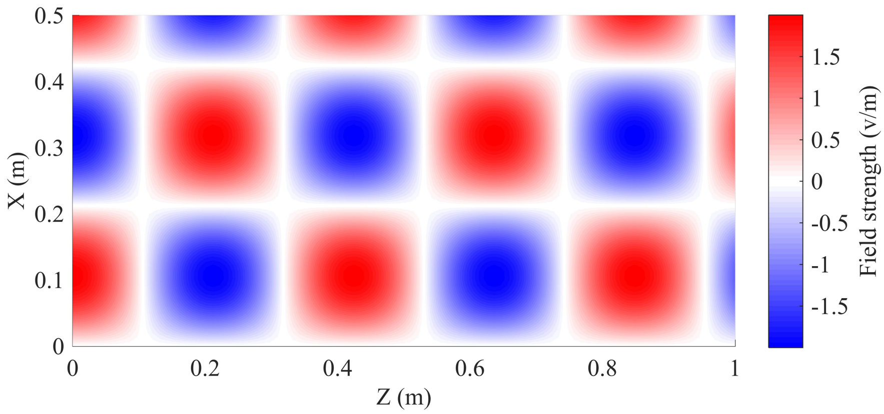
[4] Essentially this is a 'snapshot' of the time-varying field at $t = 0$.



(a)



(b)



(c)

Fig. 3: Real part of the electric field for a 1 GHz wave above a PEC ground plane (at $x = 0$), with $\theta = 45°$: (a) incident wave; (b) reflected wave; and (c) total field.

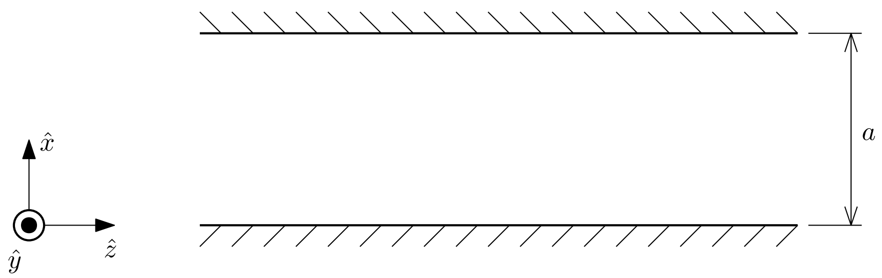


Fig. 4: Transmission line formed by two parallel plates separated by distance a.

ponent of the electric field must go to zero, thus at $x = a$

$$\sin(ka\cos\theta) = 0 \tag{17}$$

this expression is zero when

$$ka\cos\theta = \pi, \tag{18}$$

leading to

$$\cos\theta = \frac{\pi}{ka} \tag{19}$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{\pi}{ka}\right)^2}. \tag{20}$$

The fields in the parallel plate transmission line (for the fundamental mode) are thus

$$\mathbf{E} = E_0 2j\sin\left(\frac{\pi x}{a}\right)\exp\left\{-jkz\sqrt{1-\left(\frac{\pi}{ka}\right)^2}\right\}\hat{a_y} \tag{21}$$

$$\begin{aligned}\mathbf{H} =& -\frac{E_0}{\eta_0}\sqrt{1-\left(\frac{\pi}{ka}\right)^2}\,2j\sin\left(\frac{\pi x}{a}\right)\exp\left\{-jkz\sqrt{1-\left(\frac{\pi}{ka}\right)^2}\right\}\hat{a_x}\\ &-\frac{E_0}{\eta_0}\frac{\pi}{ka}\,2\cos\left(\frac{\pi x}{a}\right)\exp\left\{-jkz\sqrt{1-\left(\frac{\pi}{ka}\right)^2}\right\}\hat{a_z}.\end{aligned} \tag{22}$$

Note:

- E_y and H_x (the transverse components, i.e., orthogonal to the direction of propagation, $\hat{z}$) are in phase, but not with H_z.
- E_y has a $\sin\left(\frac{\pi x}{a}\right)$ variation.
- We choose θ to give the first null, we could have chosen others—there are an infinite number of TE modes that could be supported!
- We could rework the entire analysis for parallel polarisation, leading to transverse magnetic (TM) modes.

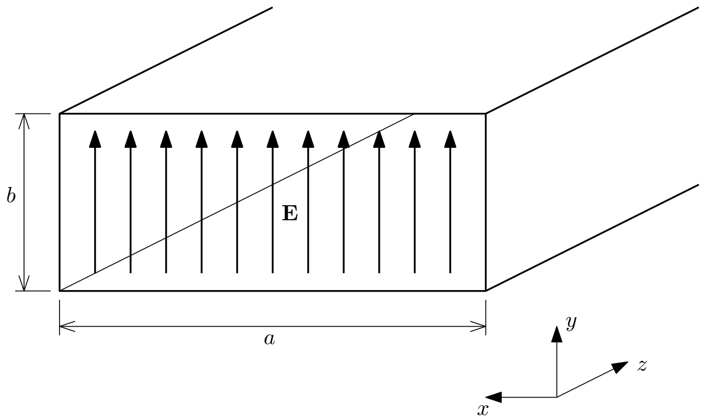


Fig. 5: Electric fields within a rectangular waveguide with dimensions $a \times b$.

4 Rectangular Waveguide

In the parallel plane transmission line $\mathbf{E}$ is orientated in the $\hat{y}$ direction only. We could thus put in additional PEC planes at $y = 0$ and $y = b$ *without altering the fields within the waveguide.* Fig. 5 shows the orientation of the electric field within the waveguide, in particular, we note that the field is *only* tangential to the PEC planes at $x = 0$ and $x = a$, however, for the PEC planes at $y = 0$ and $y = b$ the field is normal (and there is no boundary condition that needs to be satisfied).

4.1 $\mathbf{TE_{10}}$ Field Components

Based on the expressions for the parallel plate transmission line, the field components for the fundamental mode (termed the TE_{10} mode) within a rectangular waveguide with dimensions[5] $a \times b$ are

$$\left.\begin{aligned} E_y &= E_0 \sin\left(\frac{\pi x}{a}\right) \\ H_x &= -\frac{E_0}{Z_{TE}} \sin\left(\frac{\pi x}{a}\right) \\ H_z &= \frac{j E_0 \pi \cos\left(\frac{\pi x}{a}\right)}{\eta_0 k a} \end{aligned}\right\} \cdot \exp\left\{-jkz\sqrt{1-\left(\frac{\pi}{ka}\right)^2}\right\} \tag{23}$$

where

$$Z_{TE} = \frac{\eta_0}{\sqrt{1-\left(\frac{\pi}{ka}\right)^2}} \tag{24}$$

. Points to note:

- For the wave to propagate the $\left\{-jkz\sqrt{1-\left(\frac{\pi}{ka}\right)^2}\right\}$ term must be complex. However, the term in the square root is a function of wavelength (and thus of the frequency). There are three possible cases:

[5]By convention $a > b$.

(i) $ka > \pi$: $1 - \left(\frac{\pi}{ka}\right)^2$ is positive (wave will propagate).

(ii) $ka < \pi$: $1 - \left(\frac{\pi}{ka}\right)^2$ is negative (wave is rapidly attenuated, why?).

(iii) $ka = \pi$: $1 - \left(\frac{\pi}{ka}\right)^2 = 0$ (this termed the cut-off point).

- $\frac{E}{H}$ is not a constant (as it is for a TEM wave), and varies across the width of the waveguide. Hence the impedance is not uniquely defined (and in any case is also going to be a function of the frequency).

4.2 Cutoff

Waveguides have a *high-pass* characteristic: when the wavelength is larger than twice the largest dimension (i.e., $2a$) the PEC boundary conditions cannot be satisfied and the wave will not propagate. The derivation for the cutoff frequency is

$$\begin{aligned} 1 - \left(\frac{\pi}{ka}\right) &= 0 \\ \frac{\lambda_c \pi}{2\pi a} &= 1 \\ \lambda_c &= 2a \end{aligned} \tag{25}$$

and since $f_c \lambda_c = v$

$$f_c = \frac{v}{2a}. \tag{26}$$

Hence for $f > f_c$, the term $\exp\left\{-jkz\sqrt{\cdot}\right\}$ will be complex and the wave will propagate. While for $f < f_c$ the term $\exp\left\{-jkz\sqrt{\cdot}\right\}$, is real, resulting in rapid attenuation (i.e., the mode is cutoff).

4.3 Higher Order Modes

Waveguides are typically operated so that only one mode dominates (in most cases this is the fundamental, i.e., TE_{10}). As the frequency increases higher order modes are possible—the field expressions can be found by observing that in (17) the boundary condition can also satisfied for

$$ka \cos\theta = n\pi \tag{27}$$

where $n = 1 \ldots$ Carrying through this analysis leads to the following expression for the cutoff frequencies of the TE_{n0} mode

$$f_c = \frac{nv}{2a}. \tag{28}$$

For a typical waveguide we operate in the frequency range above cutoff for the TE_{10} mode, but *below* the TE_{20} cutoff frequency.